ผลเฉลยเชิงวิเคราะห์ของสนามแม่เหล็กจากแหล่งจ่ายไฟฟ้ากระแสตรง ซึ่งฝังในตัวกลางสองชั้นที่สภาพนำไฟฟ้าแปรผันแบบเลขชี้กำลัง

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บทคัดย่อ

ผลเฉลยเชิงวิเคราะห์ของสนามแม่เหล็กในสภาวะคงตัวอันเนื่องมาจากแหล่งกำเนิดไฟฟ้ากระแสตรงได้ถูกพัฒนาขึ้นสำหรับปัญหาของแหล่งจ่ายไฟฟ้าและตัวรับสัญญาณซึ่งถูกฝังอยู่ในพื้นผิวดินที่มีลักษณะเป็นชั้นขนานกับแนวระดับ ตัวแบบของพื้นโลกที่มีลักษณะเป็นชั้น แบบ 2 ชั้นต่อเนื่องกันถูกนำมาพิจารณาในกรณีที่สภาพนำไฟฟ้าเปลี่ยนแปลงแบบเลขชี้กำลังตามความลึกของพื้นโลก การแปลงฮันเกลแบบทั่วไปถูกนำมาใช้ในปัญหาซึ่งทำให้ได้ผลเฉลยเชิงวิเคราะห์ ผลลัพท์ที่เกิดขึ้นถูกนำมาเขียนกราฟแสดงพฤติกรรมของสนามแม่เหล็ก แนวเส้นโค้งของสนามแม่เหล็กแสดงถึงนัยสำคัญบางประการของการแปรผันของสภาพนำไฟฟ้า

คำสำคัญ: สภาพต้านทานแบบแมกนิโทเมตริก สภาพนำไฟฟ้าแปรผันแบบเลขชี้กำลัง การแปลงฮันเกล

Analytical Solution of Magnetic Field from a DC Source Buried in a Two-Layered Exponentially Varying Conductive Medium

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Abstract

An analytical solution of the steady state magnetic field due to a direct current point source is developed for the problem of a buried current source and a buried receiver within a horizontally stratified layered earth structure. A two-layered continuously earth model is considered in the case of an electrical conductivity varies exponentially with depth. The generalized Hankel transform is applied to our problem and then analytical result is obtained. The magnetic field results are plotted to show the behavior of the fields. The curves of magnetic fields show some significance to the variation of conductivity.

Keywords: magnetometric resistivity, exponentially varying conductivity, Hankel transform

1. Introduction

Usually interpretations of traditional resistivity soundings are conducted by assuming that the earth's structure consists of conductive horizontally stratified layers. Many authors have investigated the nature of the resistivity response resulting from a horizontally stratified layered earth whose electrical conductivity varies exponentially with depth. Stoyer and Wait [1] firstly considered an exponential earth structure under homogeneous overburden. Banerjee et al. [2], Kim and Lee [3], Sripanya and Yooyuanyong [4] studied the problem of a multilayered earth with layers having exponentially varying resistivities and derived the specific case for a two-layered model. Sripanya [5] solved the problem with a buried electrode of magnetometric resistivity sounding in a heterogeneous medium beneath a homogeneous overburden. A host medium has electrical conductivity varying exponentially with depth.

In this paper, we develop an analytical solution of the steady state magnetic field resulting from a direct current point source buried within a two-layered heterogeneous earth structure with each layer having exponentially varying conductivity. The generalized Hankel transform is applied to our problem and then analytical result is obtained.

2. Model and Basic Equations

Let us consider a geometric model of the earth's structure described and discussed by Sato [6]. The earth's structure consists of two conductive half-spaces (see Figure 1). The half-space above the ground surface $\left(z<0\right)$ is a region of air, denoted by layer 0. The half-space below the ground surface $\left(z>0\right)$ is a two-layered horizontally stratified earth with depth to the layer h_1 measured from the ground surface. Electrical conductivity in each conductive layer is a function of only depth z, i.e., $\sigma_k\left(z\right)$ for layer $0\le k\le 2$. A buried current source is located at the boundary interface $z=h_1$ of layer 1 and layer 2.

2.1 Magnetic Field from a Direct Current Source

The Maxwell's equations in the spatial frequency domain can be used to find out the steady state magnetic field for this problem (Edwards [7], Sripanya [8]). The problem is symmetric around an axis, in cylindrical coordinates, the magnetic field vector has only the azimuthal component. We use the function H to explain the azimuthal component of the magnetic field in the following equations and we then have

$$\frac{\partial^2 H}{\partial z^2}+\sigma\frac{\partial}{\partial z}\left(\frac{1}{\sigma}\right)\frac{\partial H}{\partial z}+\frac{\partial^2 H}{\partial r^2}+\frac{1}{r}\frac{\partial H}{\partial r}-\frac{1}{r^2}H=0. \quad (1)$$

The Hankel transform (Ali and Kalla [9]) is used to solve the problem, namely

$$\tilde{H}\left(\lambda,z\right)=\int_0^\infty \lambda r H\left(r,z\right)J_1\left(\lambda r\right)dr \quad \text{and} \quad H\left(r,z\right)=\int_0^\infty \tilde{H}\left(\lambda,z\right)J_1\left(\lambda r\right)d\lambda, \quad (2)$$

where J_1 is the Bessel function of the first kind of order one. Applying the generalized Hankel transform to equation (1), we obtain

$$\frac{\partial^2 \tilde{H}}{\partial z^2}+\sigma\frac{\partial}{\partial z}\left(\frac{1}{\sigma}\right)\frac{\partial \tilde{H}}{\partial z}-\lambda^2\tilde{H}=0. \quad (3)$$

Therefore, the steady state magnetic field in each conductive layer can be obtained by applying the inverse generalized Hankel transform as shown in equation (2) to the solution of equation (3).

2.2 Boundary Conditions

Our problem satisfies the following boundary conditions:

1. The magnetic field $\tilde{H}_0$ certainly converges to zero as z tends to minus infinity.

2. The magnetic field should be continuous on the boundary interfaces except on the interface located to electrode, i.e.,

$$\lim_{z\to h_0^-}\tilde{H}_0=\lim_{z\to h_1^+}\tilde{H}_1. \quad (4)$$

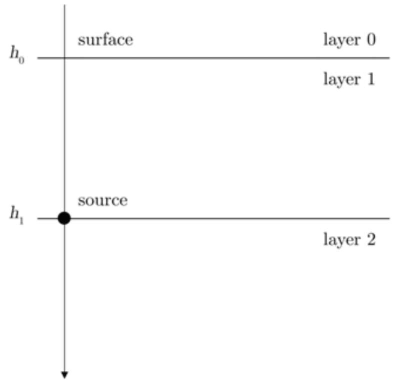

Figure 1 Geometric model of the earth's structure.

3. The radial component of a vector of the electric field should also be continuous on the boundary interfaces (Chen and Oldenburg [10]), i.e., for each $0 \le k \le 1$,

$$\lim_{z \to h_k^-} \frac{1}{\sigma_k} \frac{\partial \tilde{H}_k}{\partial z} = \lim_{z \to h_k^+} \frac{1}{\sigma_{k+1}} \frac{\partial \tilde{H}_{k+1}}{\partial z}. \quad 5)$$

4. The current intensity should be equal to the total current flowing through any surface of the cylinder around the current source (Sato [6]), i.e.,

$$2\pi \lim_{h \to 0} \left(\tilde{H}_2 \Big|_{z=h_1+h} - \tilde{H}_1 \Big|_{z=h_1-h} \right) = I. \quad (6)$$

5. The magnetic field $\tilde{H}_2$ certainly converges to zero as z tends to infinity.

3. Solution of the Problem

The conductivity variation of an exponentially varying conductive layer k of the ground is denoted by

$$\sigma_k(z) = a_k \exp\left(b_k \left(z - h_{k-1}\right)\right), \quad (7)$$

where a_k, b_k and h_{k-1} are constants that preserve $\sigma_k(z) > 0$. Hence, the equation for the magnetic field in each conductive layer can be simplified by substituting equation (7) into (3) and we obtain

$$\frac{\partial^2 \tilde{H}_k}{\partial z^2} - b_k \frac{\partial \tilde{H}_k}{\partial z} - \lambda^2 \tilde{H}_k = 0. \quad (8)$$

Solving the above equation yields

$$\tilde{H}_k(\lambda, z) = A_k \exp\left(\left(z - h_{k-1}\right)\xi_k\right) + B_k \exp\left(\left(z - h_{k-1}\right)\zeta_k\right), \quad (9)$$

where ξ_k and ζ_k are given by

$$\xi_k = \frac{1}{2}\left(b_k - \delta_k\right), \quad \zeta_k = \frac{1}{2}\left(b_k + \delta_k\right) \quad \text{and} \quad \delta_k = \left(b_k^2 + 4\lambda^2\right)^{1/2}. \tag{10}$$

The coefficients A_k and B_k are the unknown constants, which can be specified by applying the boundary conditions as shown in Section 2.2.

3.1 Two-layered Earth Model

Let us consider a two-layered electrically conductive earth structure with a nonconductive medium of layer 0 that represents an air region whose conductivity is approximately equal to zero. An overburden medium, denoted by layer 1, has an exponentially varying conductivity $\sigma_1\left(z\right)$ with thickness h_1, whereas a host medium, denoted by layer 2, also having an exponentially varying conductivity $\sigma_2\left(z\right)$. A point source of direct current I and a buried receiver are located at the boundary interface $z = h_1$. The steady state magnetic field in a host medium (layer below source and receiver) can be written as

$$H\left(r,z\right) = \int_0^\infty \frac{I}{2\pi}\left(\frac{\left(\xi_1 - \zeta_1 \exp\left(h_1\delta_1\right)\right)\exp\left(\left(z-h_1\right)\xi_2\right)}{\xi_1 - \xi_2 - \left(\zeta_1 - \xi_2\right)\exp\left(h_1\delta_1\right)}\right)J_1\left(\lambda r\right)d\lambda. \tag{11}$$

3.2 Uniform Half-space Model

The steady state magnetic field in a homogeneous ground can be determined by reducing equation (11) and then we obtain

$$H\left(r,z\right) = \frac{I}{4\pi r}\left(2 - \frac{z+h_1}{\sqrt{r^2 + \left(z+h_1\right)^2}} - \frac{z-h_1}{\sqrt{r^2+\left(z-h_1\right)^2}}\right), \tag{12}$$

which is the same as obtained by Sripanya [8].

4. Numerical Experiments

In our sample tests, we calculate the steady state magnetic fields resulting from two models of practical interest. Both of the example models are two-layered electrically conductive earth structures with a nonconductive layer 0, as shown in Section 3. These models are referred to as Models 1 and 2, which are given in equation (11). The values of the model parameters are tabulated in Table 1. Chave's algorithm [11] is used for numerically calculating the inverse Hankel transform of the magnetic field solutions. The electric current of 1 ampere is used in our computations. The results of our models are plotted and compared to show the behavior of magnetic fields against source-receiver spacing r at different depths $z = 5, 5.5, \ldots, 15$ metres as shown in Figure 2. The graphs show certainly that the magnetic field intensity is inversely varying with respect to the depth and source-receiver spacing. We have seen further that the curves of magnetic fields obtained from the first model are quite different from the second model. The contrast of electrical conductivity gives effect to the magnitude of magnetic field. It can be seen that the magnitude of magnetic field from the first model is much higher than the magnetic field from the second model at the same depth and source-receiver spacing. We observe that if the conductivity of a boundary layer is high, it will lead to the large sized magnitude of magnetic field. This means that the differences of magnetic curves are depended on the variation of conductivity.

Table 1 Model parameters used in our sample tests.

Model	a (S/m)	b_1 (m^{-1})	b_2 (m^{-1})	h_1 (m)
1	0.1	0.25	0.1	5
2	0.5	– 0.25	– 0.1	5

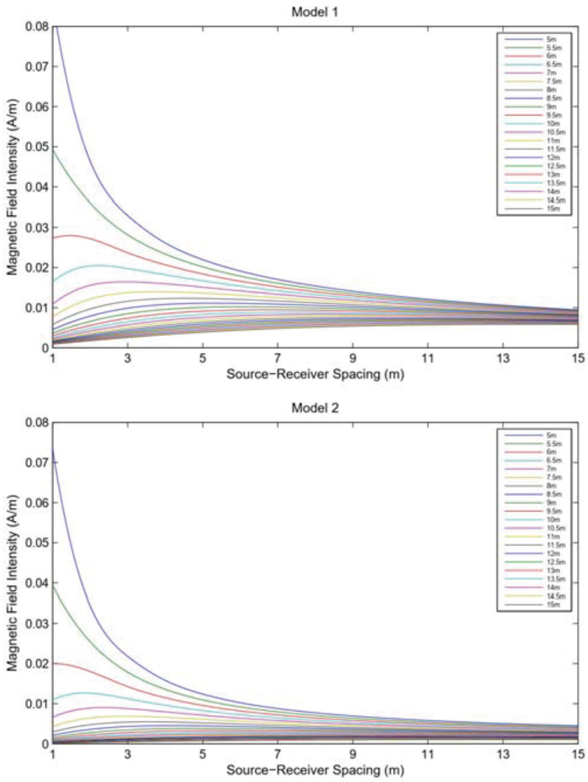

Figure 2 Behavior of magnetic fields from our example models.

5. Conclusions

We have derived an analytical solution of the steady state magnetic field for the problem of source and receiver electrodes buried within a two-layered stratified earth with exponentially varying conductive overburden and host medium. The generalized Hankel transform is applied to our problem and then analytical result is obtained. The effects of magnetic fields are plotted and compared to show the behavior in response to different ground structures at many depths while some parameters are approximately given. The curves of magnetic fields show some significance to the variation of conductivity. The solution can be used to interpret downhole and marine MMR data in which geophysical inversion is required (see Sripanya [12]).

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